Perihelion Precession of Mercury, full Calculation

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In reality, complete calculation of the perihelion precession of Mercury takes 50" in the century.

43" is apparent in the reference system of Mercury. This allows to omit the Lorentz factor in the energy law:

$$E = \frac{mv^2}{2} - G\frac{Mm}{r} - GM\frac{L^2}{mc^2r^3}\,.$$

But in reality, the value of 43" is measured for the reference system of the sun. Therefore relativistic kinetic energy

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and relativistic angular momentum

$$L = \frac{m[\vec{r} \times \vec{v}]}{\sqrt{1 - \frac{v^2}{c^2}}}$$

necessarily appertain to the energy law:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - G\frac{Mm_0}{r} - \frac{GML^2}{m_0 c^2 r^3\sqrt{1 - \frac{v^2}{c^2}}}$$

Such an equation to solve, however, must be incredibly difficult. Therefore we assume that the equation of general relativity with r^{-3}

-Term already supplies 43" and we calculate only addition without r $^{-3}$
-Term. Then the energy law looks:

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - G\frac{Mm_0}{r}.$$

The Lorentz factor we can replace through the angular momentum and then place them in the energy equation:

$$L = \frac{m_0 r^2 \dot{\phi}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \dot{\phi} = \frac{L}{m_0 r^2}\sqrt{1-\frac{v^2}{c^2}}$$

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = r'\dot{\phi}$$

$$\frac{v^2}{c^2} = \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2} = \frac{(r'^2 + r^2)\dot{\phi}^2}{c^2} = \left(\frac{L}{m_0 c}\right)^2\left(\frac{r'^2}{r^4} + \frac{1}{r^2}\right)\left(1-\frac{v^2}{c^2}\right)$$

With the new variable

$$s = \frac{1}{r},\ s' = -\frac{r'}{r^2}$$

we get

$$\frac{v^2}{c^2} = \left(\frac{L}{m_0 c}\right)^2 \left(s'^2 + s^2\right)\left(1-\frac{v^2}{c^2}\right)$$

and further

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{1+\left(\frac{L}{m_0 c}\right)^2 \left(s'^2+s^2\right)}\ .$$

This term we use now in the energy equation:

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - G\frac{Mm_0}{r} = m_0 c^2 \sqrt{1+\left(\frac{L}{m_0 c}\right)^2 \left(s'^2+s^2\right)} - GMm_0 s\ .$$

It follows

$$\left(E + GMm_0 s\right)^2 = m_0^2 c^4 \left[1+\left(\frac{L}{m_0 c}\right)^2 \left(s'^2+s^2\right)\right]$$

$$E^2 + 2EGMm_0 s + (GMm_0)^2 s^2 = m_0^2 c^4 + L^2 c^2 s'^2 + L^2 c^2 s^2$$

$$s'^2 = \left(\frac{E}{Lc}\right)^2 - \left(\frac{m_0 c}{L}\right)^2 + \frac{2EGMm_0}{L^2 c^2} s - s^2 \left[1-\left(\frac{GMm_0}{Lc}\right)^2\right].$$

Now we use

$$\lambda^2 = \left[1-\left(\frac{GMm_0}{Lc}\right)^2\right]$$

and it follows

$$\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{E}{Lc}\right)^2 - \left(\frac{m_0 c}{L}\right)^2 + \frac{2EGMm_0}{L^2 c^2} s - s^2 \lambda^2$$

$$\lambda d\phi = \frac{ds}{\sqrt{\left(\frac{E}{Lc\lambda}\right)^2 - \left(\frac{m_0 c}{L\lambda}\right)^2 + \frac{2EGMm_0}{L^2 c^2 \lambda^2} s - s^2}}$$

The Swing from r_{min} to r_{max} and back supplies of the right side 2π. It follows

$$\lambda(\phi - \phi_0) = 2\pi, \phi - \phi_0 = \frac{2\pi}{\lambda}.$$

Thus, the perihelion shift in the direction of rotation (figure below) is equal:

$$\Delta\phi = \frac{2\pi}{\lambda} - 2\pi = 2\pi(\lambda^{-1} - 1) =$$

$$= 2\pi\left[\frac{1}{\sqrt{1-\left(\frac{GMm_0}{Lc}\right)^2}} - 1\right] \approx \pi\left(\frac{GMm_0}{Lc}\right)^2$$

This is per cycle $\Delta\phi \approx 8.35\cdot 10^{-8}\, rad$ and per century about 7".

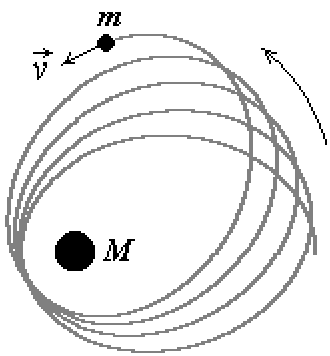

Figure. The perihelion shift in the direction of rotation.

In this way according to relativity theory complete calculation of the perihelion precession of Mercury takes 50" instead of 43" in the century.

Literature

Norbert Dragon. Geometrie der Relativitätstheorie.
http://theory.gsi.de/~vanhees/faq/relativity/

Full text in Germany: http://www.walter-orlov.wg.am/buch/